Design and Analysis of Algorithms, Chennai Mathematical Institute
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Week - 07
Module - 03
Lecture - 46
Grid Paths

So, continuing our discussion about efficiently computing recursive functions, let us look at the problem of computing grid paths.

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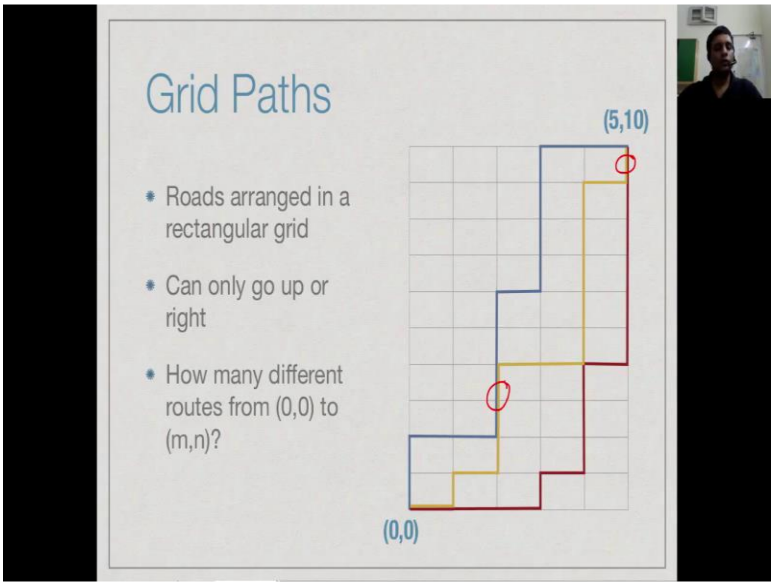

So, we have a grid, a rectangular grid, where we start walking at the bottom left corner, and the rule is that we can only go up or right. So, we want to start at the bottom and we want to reach the top right corner. So, we can number the coordinates. So, the bottom right corner we call (0,0). This particular grid has got 5 columns and 10 rows. So, the top right corner is (5, 10). And the question that we ask is how many different ways are going, are that go from (0, 0) to (5, 10).

So, what do we mean by different ways. Well, here for instance is one grid path. This blue line takes us up from (0, 0), then right, then up, then right and so on. So, this traces out one particular sequence of edges along this grid taking us from the bottom left corner to the top right. So, we could, of course choose a different one, which in this particular one the red one and the blue one are completely disjoint. They do not use any of the same edges, but in general, I could have paths which overlap with the other ones. So, this yellow one partly overlaps with the blue at the middle, over here, and then it overlaps with the red one over here, right. We want to know how many such different paths are there from (0, 0) to (m, n).

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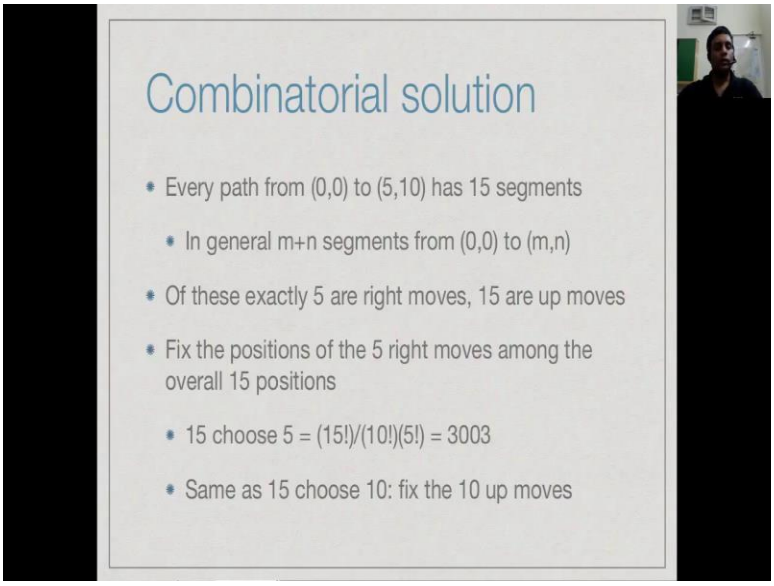

Now, it turns out this problem is actually a very classical problem in combinatorics counting. So, the way to analyze this is to see, that if I want to go from the bottom to the top, then I must, on the, in one-dimensional I must go from 0 to 5. In the other dimensions, I must go to 0 to 10. So, totally I must make 15 steps, right. There is no choice, I must walk 15 segments.

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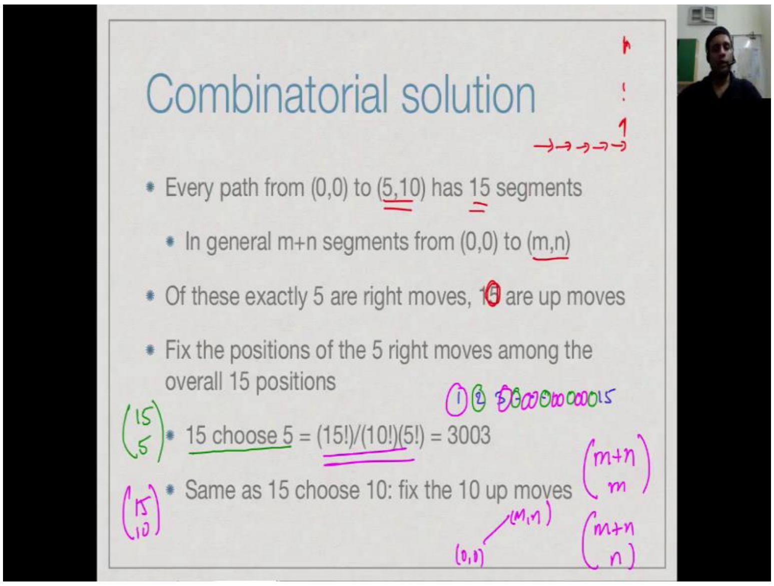

In general, if I am going to some (m, n), I must walk m plus n segments. Now, if I have these 15 segments and I am going to (5,10), that means, I have to go right, 5 times, right, and I have to go up 10 times. Now, in which sequence I do these rights and ups is what determines which path I take, right. But every path will have exactly 5 right moves and exactly 10 up moves, right. So, there will be 10 up moves and 5 right moves.

So, now if I take this and I think of this as an overall thing saying this is my 1st move, this is my 2nd move, this is my 3rd move and so on, up to the 15th move. Then if I tell you that you moved right, I moved right at these positions, at some five positions, then automatically I must have moved up at the remaining positions because I have to do exactly 5 and exactly 5 right and 10 up. So, therefore, all I need to do to determine exactly which path I am taking is to fix the position of the 5 rights moves among the total 15, right. So, among 15 positions I choose 5. This is usually written as 15 choose 5, right. So, this is a very standard, combinatorial thing, 15 choose 5 n choose k is n factorial divided by k factorial into n minus k factorial, right. So, 15 factorial divided by 10 factorial times 5 factorial, it happens to be 3003 ((Refer Time: 03:24)). So, for this particular grid there are 3003 ways.

Now, of course, instead of choosing the right positions I could also have told you the 10 positions where I moved up, that leaves 5 open positions where I ((Refer Time: 03:39)). So, this would give us 15 choose 10, and so it is not a coincident that 15 choose 10 and

15 choose 5 are in fact the same expression. So, whether you choose to compute it as 15 choose 5 or 15 choose 10, it does not matter. So, in general it is going to be m plus n choose m or m plus n choose n where I am going from (0, 0) up to (m, n), right. I have to make m right moves and n up moves. So, I need to choose m out of m plus n total moves or n out of m plus n, both of them will give me the same expression, because that is why, n choose k, n choose k is n factorial k factorial n minus k. So, if you just exchange k and n minus k you get the same expression.

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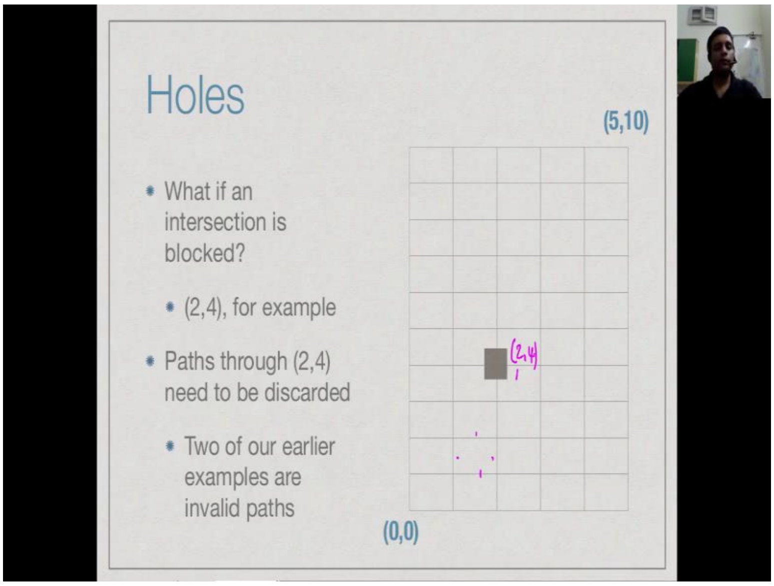

So, that is all very well, but what, for example, if this is not a perfect grid. Supposing we have some intersections, which are blocked. So, in this particular case, if you look at this intersection, which is (2, 4), right, it is 1, 2 and then 1, 2, 3, 4. So, we have put a black mark to indicate that for the moment, you cannot go through it, right. So, any part that goes through (2, 4), should not be counted among the valid path from (0, 0) to (5, 10).

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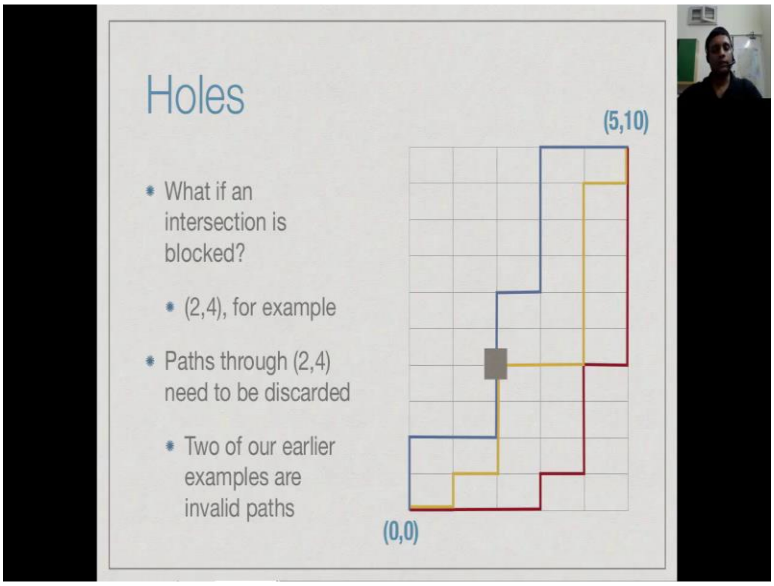

So, the blue path that we had drawn before actually goes through this intersection. So, this path is no longer a valid path. The red path is ok, because it bypasses it, but the yellow path unfortunately also goes through this intersection. So, of the three paths, that we had seen so far, two actually do not go through. So, the question now is, out of those 3003 paths that we said were there, from (0, 0) to (5, 10), how many of them are still valid if you are not allow to go through this intersection.

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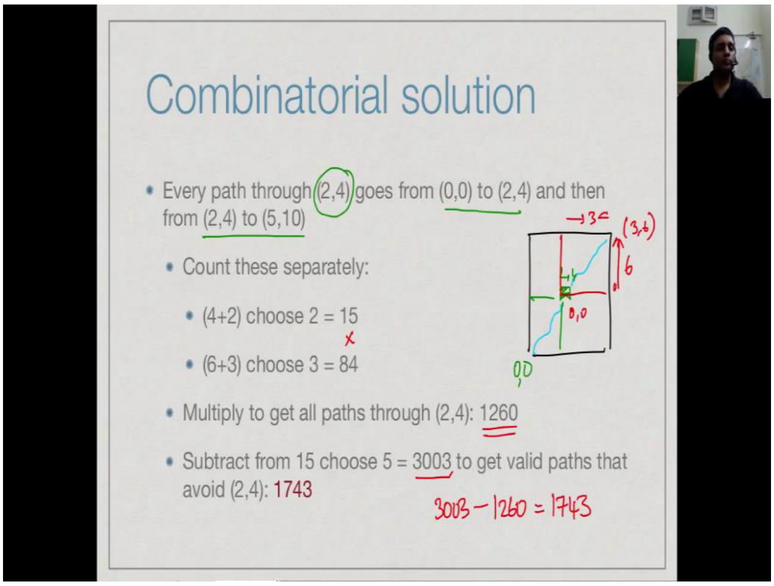

So, it turns over that actually this also has a combinatorial solution. So, what you can say is that in order to go from here and if I want to avoid a block intersection. So, let me see how many ways are there of going through it and just remove them, right. So, what I will do is, I will say, that every path it goes through the current block intersection is a path from (0, 0) to 92, 4) followed by a path from (2, 4) to (5, 10) because it goes through (2, 4). So, we can think of this as a smaller grid from (0, 0) to (2, 4), right.

So, if we solve, this is, this general m plus n m plus n choose n, right. So, I get 6 choose 2. So, I get there are 15 ways to go from (0, 0) to (2, 4). Likewise, if I consider this grid, then this is 3 and this is 6 because it was 10 and 5, I have done 2 and 4 respectively. So, after 2 there is still 3 left horizontally; after 4 there is still 6 left vertically. So, this is like going from a new (0, 0) to (3, 6), right. So, there I get 6 plus 3 choose 3 and this turns out to be 84.

Now, any part, which comes to the bottom 2 to 4 followed by any part, that goes from there to the top is a valid path passing through 2 to 4. So, I multiply these two numbers, I have to take 15 times 84 and again 1260. So, this is a total number of paths, which are going through (2,4), but I do not want paths going through (2,4). I am saying, that paths going 2 to 4 are not allowed. So, I must count all these parts as invalid. So, I take the original 3003 subtract. So, I take 3003 and I subtract 1260 and I get 1743, right. So, taking the combinatorial exercise to the next step. I can find out how many paths go from the origin to the given point on the top right provided one path one position is blocked.

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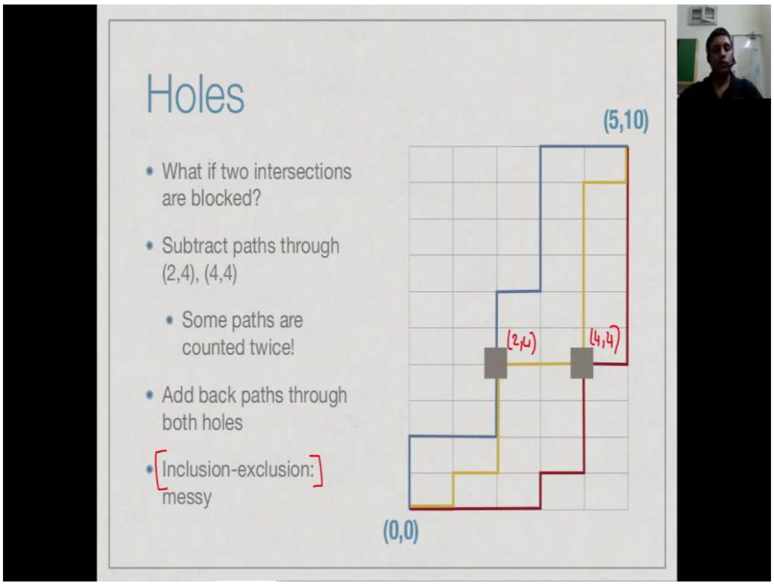

So, this imperfection could be more complex. I could have two positions blocked, right. In this case, the blue, yellow and red paths are all invalid because the red path also happens to get blocked. So, now I have blocked at (2,4) and also at (4,4), right. So, if I count all the paths going through two, (2,4), which I have already done, I can subtract this. Similarly, I can compute all the paths going through (4,4) and subtract those.

But now what happens is, this yellow path is subtracted twice because it is part of the paths going through (2,4) and part of the paths going through (4,4), right. So, I have accidentally removed it twice from my total, so I have to put it back. You have to now compute those paths, which go through both the intersections and add them back and this combinatorial is called inclusion and exclusion. You come across it also sometimes when you do Venn diagram, when you do sets.

If you want to find out, now you say, how many people are, how many sets have one element, how many sets are ((Refer Time 08:33)) intersection, how many sets you have, you know, you are, students taking three subjects, English, history and physics, and then so many taking English and history, so many taking history and physics and so on. So, when you do that kind of counting you will exactly do this inclusion exclusion, right. So, as we get more and more messy grids, this combinatorial question becomes more and more complicated to solve this way.